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On the Resonance Effect in a Storm Surge (Part I)

By Hikoji YAMADA, Jun-ichi OKABE[1], and Masako KUMAZAWA[2]

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Abstract

More or less varied estimations seem to be asserted concerning the sudden growth of high water as observed when a pressure disturbance passes through the resonance point defined with respect to the depth of water. Computations are performed in this paper with a view to studying the theoretical phase of the problem. Confining ourselves to the case of one-dimensional expanse of water surface, the equations of motion are reduced to the forms of the so-called 'shallow water theory'. In this simplification, the non-linear terms of the equations are preserved, while, on the other hand, effects of viscosity are altogether left out of consideration, because, as pointed out in the second report, they possibly do not modify the situation drastically. In the first place, the steady swell accompanying a disturbance of invariable intensity which advances with a constant velocity upon the surface of water of a uniform depth is formulated analytically without difficulty, and when the state is of near resonance results of the linear approximation generally used in the past are shown to need large corrections. In the next place, the swell of water taking place when a disturbance, climbing up an inclined bed and approaching a shore, goes through the resonance point is calculated numerically by the aid of an electronic computer. It is found that the smaller the slope of the bottom the higher is the tidal level, and that even an inclination usually experienced is still sufficient to produce an ascent of level up to several tens percent. It is concluded, therefore, that the resonance phenomenon could not possibly be neglected, except when it is canceled out by chance owing to some three-dimensional influences.

1. Introduction

If the velocity V with which an atmospheric disturbance advances upon the surface of water approaches the value $(gh)^{1/2}$, the propagation velocity of the tidal wave, h and g being the depth of water in that region and the acceleration due to gravity respectively, then deformation (elevation or depression) of the surface from the horizontal is enlarged markedly. This fact has been generally accepted in understanding miscellaneous phenomena in coastal oceanography; see, for example, (1-5)[3]. However, quantitative discussions of this problem in connection with topographical features are very few, and in fact none of them may be available for theoretical estimations. Besides, on reflection that a computation was once published (6) that in a case of two-dimensional surface the resonance was not felt in the least being masked by other environments, the effect is known to be greatly influenced by topographical conditions.

On the other hand, inducing quantitative data successfully out of observed

1) Professor, Research Institute for Applied Mechanics, Kyushu University.

2) Former Research Assistant, Faculty of Engineering, Kyoto University.

3) As to the numerals in parentheses, see the list of references at the end of the paper.

facts would be obstructed in general by difficulties, resulting from complications involved in natural phenomena, encountered in extracting resonance effect in its pure form out of entangled integrity. Fluctuation of the water level caused in Lake Michigan by the transit of an almost theoretical pressure jump front was one of these rare cases (4, 5), nevertheless under natural topographical and weather conditions it was extremely difficult to remove all ambiguities in order to arrive at the resonance effect itself. Ideal conditions might be realized only by means of experiment. Although this must be of course a very effective method of research (7), we are not yet in a position to draw satisfactorily reliable numerical data through experimentation. Under these circumstances, mathematical analysis can be put to use most fruitfully, and this is the background of the so-called mathematical hydraulics; besides recent development of electronic digital computers has guaranteed the carrying out of complicated computations. This is the approach taken in this paper.

Now, since our object is primarily to evaluate the resonance effect free from all influences of environment, initial, as well as boundary, conditions have to be assumed in such forms that inappropriateness inherent possibly in them would not result in a large error in the calculated surface elevation. As the initial condition therefore we need to take a steady wave upon a uniform depth, an atmospheric disturbance being supposed to enter into the field of an inclined bottom accompanied by this wave pattern. It would be quite natural to assume that the area is bounded by a shore at the opposite end, but with a special view to excluding from the resonance wave height those components which result from the reflection of waves by the shore, we shall assume in the following that the region in question is extended indefinitely by shallow water of a uniform depth.

Some considerations would be necessary for the differential equations to be solved under the initial, and the boundary, conditions mentioned above. Since we are concerned with resonance phenomena, we should expect that the linear approximation ceases to be valid just as it does in the resonance taking place in one-dimensional oscillation of a particle, and by the analogy with this case neglect of viscosity might similarly be concluded misleading. Although this point of view is quite correct as a matter of fact for a steady wave traveling on a uniform depth, for a sloping bottom, on the other hand, the condition of resonance, or of quasi-resonance, can be maintained only for a limited time, and we may expect with some certainty that the inadequate treatment of the equations during this short interval of time would not bring appreciable error in computation of tidal level under natural circumstances generally existing.

The problem of whether or not the linearization would be allowable has in fact an influence quite far-reaching. For, according to our opinion, attempts or preparations for attempts are being gradually well matured toward theoretical computations and forecasts of high waters in harbors generated by meteorological disturbances such as typhoons. While high water is produced there as a result of combined action of astronomical forces and meteorological disturbances (the depression and the driving force of wind), it may be regarded also at the same time as the synthetic effect of the in-, and out-flow through the mouth and of the mode of change of the water level proper to that harbor. So the mathematical analysis attempting to treat the problem as a whole must be undoubtedly lengthy and complicated, and accordingly absolutely inconvenient, if not impossible, to attain practical purposes. It would be preferable to linearize the differential equations governing the phenomena at a small sacrifice of accuracy, and to construct the whole surge with the aid of separate computations or estimations of constituent parts. If, therefore, there should be involved one constituent phenomenon in which non-linearity is found essential, our plan of computations should necessarily fail altogether. In this sense, the criterion is very important whether the resonance effect may be linearized without destroying the nature of the problem; as a matter of fact we have to expect in every calculation of high water practically without exception that the resonance state takes place somewhere in some water areas.

In this paper, therefore, we evaluate in the first place the rise of water surface due to resonance by treating the non-linear differential equations as they are. If this increase should not dissipate sideways owing to two-dimensional extension of the sea, it may be concluded that the resonance is quite considerable. Although the effect is assumed at first as resulting from pure depression, it will be shown in the next report that the influence of wind may also be included in the same computation scheme, or in other words, that the same type of approximate solution is valid for a meteorological disturbance as a whole. Next comes the question: The numerical values obtained through this exact procedure, how much would it be modified when we make use of linear approximation or introduce viscosity? To begin with, it is observed, for the slopes we assume, resonance computed through linearization is reduced to some extent compared with the value exactly worked out. Since, however, as mentioned before, in cases of bottoms of gradual slope, linear theory is pointed out to yield over-estimation of the effect, we may suppose that presumably linear, and non-linear, formulations give the results roughly coincident with each other for a certain small inclination of the bottom. Legitimacy of consistent use of linear approximation throughout the whole domain with complicated topography might be supported by this contemplation. In the next place, viscosity acts toward decreasing the surface elevation, as is naturally expected. But according to our estimation, modification owing to viscosity is found not so appreciable in the neighborhood of the maximum elevation. Basically speaking, viscosity effect is known to be adequately taken account of in the scope of linear theory, and the relevant terms are usually considered in computations of high waters, so that no difficult problems are presented in particular except the significance that the order of viscous damping can be estimated through our computations.

By reducing the fundamental equations into non-dimensional forms, we find that they are governed by three parameters so long as we neglect viscosity. Of these since we are most interested in the scale of a disturbance, the resonance is studied with the aid of the linearization method for a depression assumed twice as large as the preceding example. Judging from the fact that sensibly the same elevation is worked out as the result, we might safely draw the conclusion that the scale of a disturbance is not so important a factor in so far as we leave out consideration of the reserve that possibly viscosity acts in a manner somewhat differently between these two cases.

To establish the method of incorporating the data of atmospheric disturbance, variable moment by moment, constitutes a central problem in this field of research. Renewing the data at every step of computation would be not only too tedious but even unnecessary. While in practice a distribution is generally adopted, changing with time and calculated through a formula, we may conceive an alternative in which, within a certain short duration with center at a certain instant, data are assumed invariable and equal to their values at that instant. The latter, we think, may be suitable for harbors for which the formulation above-mentioned is found difficult owing to complicated environmental topography. With a view to testing the accuracy of the computation scheme in which data are to be renewed intermittently, this procedure is applied to our present example of the resonant high water. Comparing the result thus yielded with that derived from the data given continuously, it is found that the difference is quite small. This seems to be an indication that the phenomenon called 'resonance' is not in fact an event taking place at a single, isolated point, but that it should be understood as a synthetic effect over a larger domain. At the same time out of the preceding test satisfactory justification is provided for intermittent renewal of data.

These are the contents to be described in some detail in Part I and also in Part II, to be published shortly.

2. Fundamental equations and dimensional considerations

Denoting the mean position of the surface of water by $y=0$, the x-, and the y-, axes are taken in the direction of advance of the disturbance and vertically upwards respectively, t being time. The surface, and the bottom, of water are put respectively in the forms

$$y=\eta(x,t), \quad \text{and} \quad y=-h(x).$$

In this coordinate system, u, the horizontal component of the velocity of a uniform liquid, is found according to the shallow water theory (7) to satisfy the differential equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial}{\partial x}(\eta+P), \tag{1}$$

and

$$\frac{\partial \eta}{\partial t}=-\frac{\partial}{\partial x}\{u(h+\eta)\}, \tag{2}$$

where $P(x,t)$ signifies the intensity of a disturbance in the atmospheric pressure acting on the surface of the water, expressed in terms of head of a water column. Validity of the system of these fundamental equations will be assumed for the present; pertaining discussions will be found in a later section. Only it must be noted here that they hold good in so far as the boundary layer formed on the bottom be negligibly thin; in this case u may be regarded as practically uniform from the surface down to the bottom, or when expressed symbolically $u=u(x,t)$.

Since we are concerned primarily with an advancing disturbance, we write

$$P = P(X), \quad \text{where} \quad X \equiv x - Vt, \tag{3}$$

supposing it proceeds with a constant speed V without altering its interior distribution; besides what is necessary here is only such a distribution that we may put $P \equiv 0$ outside the range $-L_1 < X < L_1$. Meanwhile $h(x)$, the depth of the bottom above which this disturbance travels, is a function which starts from a certain constant value, decreases monotonically in the immediate neighborhood of $x=0$, and settles down into another positive constant. We might verify that

$$h(x) = h_0\{1 - \lambda \tanh (x/L_2)\} \tag{4}$$

is an example of these functions. It is assumed that h_0, the depth at $x=0$, coincides with the resonance depth for waves of infinitesimal amplitude defined by the equation

$$(gh_0)^{1/2} = V. \tag{5}$$

In our problem to be dealt with in the following, since the slope is traversed by a wave keeping the surface elevation η positive, the resonant point for a free wave will be displaced somewhat toward the positive side of the origin $x=0$. But it will be shown in the next section that the resonance of accompanying waves takes place exactly at the depth h_0 satisfying the relation (5), irrespective of differences of surface elevations.

Let us assume in the first place for P and h the general forms

$$P = P_0 f(X/L_1), \quad \text{and} \quad h = h_0 \varphi(x/L_2\,;\,\lambda), \tag{6}$$

where f means an arbitrary function satisfying the equalities

$$f(0) = 1, \quad \text{and} \quad f(\xi) = 0 \quad \text{for} \quad |\xi| > 1,$$

while on the other hand φ denotes another monotonic positive function subject to the conditions

$$\varphi(0) = 1, \quad \varphi(\xi) \doteqdot 1+\lambda \quad \text{and} \quad 1-\lambda \quad \text{for} \quad \xi \ll 0 \quad \text{and} \quad \xi \gg 0,$$

respectively. To be more specific, we might say that as L_1 increases the disturbance extends over larger scale and as L_2 becomes greater the bottom inclines more gently attaining to a larger distance at the same time. Choosing L_1 and h_0 as the representative lengths for the horizontal, and the vertical distances respectively, and V as the standard velocity, the various quantities can be reduced in non-dimensional forms

$$\frac{u}{V} = u_0, \quad \frac{x}{L_1} = x_0, \quad \frac{Vt}{L_1} = t_0, \quad \text{and} \quad \frac{\eta}{h_0} = \eta_0. \tag{7}$$

Finally introducing in places of the ratios of the constant lengths two parameters defined as

$$L_2/L_1 = \sigma, \quad \text{and} \quad P_0/h_0 = \alpha, \tag{8}$$

equations (1) and (2) may be transformed into

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}=-\frac{\partial \eta_0}{\partial x_0}-\alpha f'(x_0-t_0), \\ \text{and} \qquad &\frac{\partial \eta_0}{\partial t_0}=-\frac{\partial}{\partial x_0}[u_0\{\varphi(x_0/\sigma\,;\,\lambda)+\eta_0\}], \end{aligned}\right\} \tag{9}$$

when use is made of the relation (5).

As to the initial conditions in the next place, it was already mentioned that when $x_0/\sigma=\xi\ll 0$ we should have $\varphi=1+\lambda$ and that the stationary solution of equations (9) must be substituted for the initial value. It is obvious that this solution depends upon only two constants α and λ, the fact to be investigated closely in the next section. Then, provided that the functional form of $\varphi(\xi)$ be kept invariable, our solution (u_0, η_0) of high water would be governed only by α, σ, and λ, or writing explicity we have

$$u_0=u_0(x_0,\ t_0\,;\ \alpha,\ \sigma,\ \lambda), \quad \text{and} \quad \eta_0=\eta_0(x_0,\ t_0\,;\ \alpha,\ \sigma,\ \lambda).$$

In other words, if we wish to discern distinctly behaviors of high water corresponding to miscellaneous relative variations of its factors, we have to evaluate (u_0, η_0) when α, σ, and λ are supposed to alter their values in their own ways. But now we will give up the idea of those general computations, for what we have for our object in this paper are the inquiries of non-linear behaviors of a resonant high water and of possibility of linear approximation. Only one example will be mentioned later in which the effect of variation of σ is proved hardly sensible. Approximate linear dependence on α will be readily clarified if we linearize the equations.

3. Steady attendant tide on a sea of a uniform depth

If use is made of the same notations as before, except denoting the uniform depth by h_1 and expressing the vertical lengths in terms of their non-dimensional forms defined as

$$\eta/h_1=\eta_1, \quad \text{and} \quad P/h_1=p_1, \tag{10}$$

equations (1) and (2) reduce into

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial t_0}+u_0\frac{\partial u_0}{\partial x_0}=-\frac{1}{m^2}\frac{\partial}{\partial x_0}(\eta_1+p_1), \\ \text{and} \qquad &\frac{\partial \eta_1}{\partial t_0}=-\frac{\partial}{\partial x_0}\{u_0(1+\eta_1)\}, \end{aligned}\right\} \tag{11}$$

respectively, where the notation m stands for the FROUDE number of the advancing disturbance: we have from (5)

$$m=V/(gh_1)^{1/2}=(h_0/h_1)^{1/2}. \tag{12}$$

Since our attention is now concentrated specifically on a *steady attendant tide*, which means a tide accompanying steadily a moving disturbance, both of u_0 and η_1 become functions of the variable $x_0-t_0\equiv\xi$ only, and therefore equations (11) are readily integrated once, yielding

$$\left.\begin{aligned} &u_0-\frac{1}{2}u_0^2=\frac{1}{m^2}(\eta_1+p_1), \\ \text{and} \qquad &\eta_1=u_0(1+\eta_1). \end{aligned}\right\} \tag{11'}$$

If we solve the second equation with respect to u_0 and substitute it in the first, we have

$$\left.\begin{aligned} p_1 &= -\eta_1+\frac{1}{2}m^2\{1-(1+\eta_1)^{-2}\}, \\ \text{and} \qquad u_0 &= \eta_1(1+\eta_1)^{-1}. \end{aligned}\right\} \tag{13}$$

This shows that only one stream velocity u_0 is found against one surface elevation η_1 and further that the intensity of a disturbance p_1 is determined uniquely for a given value of η_1, provided the FROUDE number m is known. Several examples of those correspondences are reproduced in TABLE 1 and FIGURE 1. Inversely, however, as is clearly seen in both of them, η_1 for a given p_1 cannot only be confined to one value, but there is the maximum value of p_1, $p_1^{(m)}(\geqq 0)$ say, depending upon m, above which no values of corresponding η_1 can be found. In other words, no steady tide can possibly accompany such an atmospheric disturbance. This maximum value of p_1 is given by the solution of the equation $dp_1/d\eta_1=0$ keeping m constant, and we get

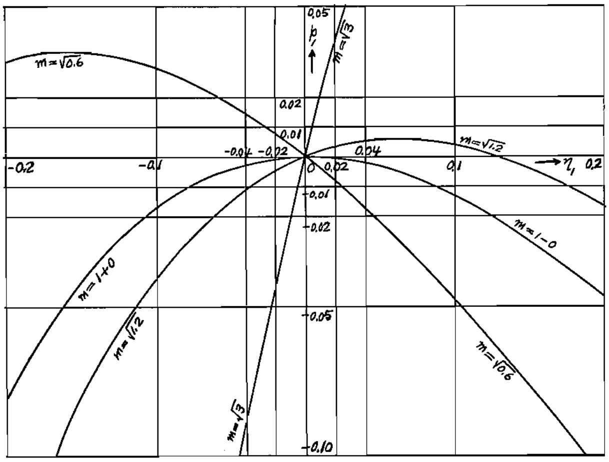

Fig. 1. Correspondence between pressure and steady water level.

$$\left.\begin{aligned} p_1^{(m)} &= 1-m^{2/3}+\frac{1}{2}m^2(1-m^{-4/3}), \\ \text{and} \qquad \eta_1^{(m)} &= m^{2/3}-1\,; \end{aligned}\right\} \tag{14}$$

in TABLE 1 these values are shown in parentheses. A continuous distribution of the surface elevation may be found for a continuous distribution of disturbance, only when a value of η_1 lying on the branch on the left or the right of $\eta_1^{(m)}$, the maximum value, which contains $\eta_1=0$ corresponds to p_1; this condition suffices to determine uniquely the value of η_1 for a given value of p_1. It may appear at first that uniqueness is still missing in case $m=1$, but it can be and readily discovered if we take into consideration the obvious fact that $\eta_1\geqq 0$ and $\leqq 0$ correspond to $m=1-0$ and $1+0$ respectively. This causes, however, at $m=1$ a sudden interchange between elevation and depression, of tidal level for a constant value of p_1. This is nothing but the mathematical ground for high water amplification as observed when the resonance point is passed over—which is the subject of this paper. Namely, in terms of the linearized model

TABLE 1.
Correspondence between pressure and steady water level.

η_1	u_0	p_1			
		$m=\sqrt{0.6}$	$m=1.0$	$m=\sqrt{1.2}$	$m=\sqrt{3.0}$
−0.40	−0.6667	−0.1334	−0.4890	−0.4890	−2.2669
−0.30	−0.4286	−0.0123	−0.2205	−0.3245	−1.2614
−0.20	−0.2500	0.0313	−0.0813	−0.1375	−0.6438
−0.18	−0.2195	0.0339	−0.0637	−0.1123	−0.5508
−0.16	−0.1905	0.0347	−0.0487	−0.0904	−0.4660
(−0.1566)	−0.1857	(0.0348)	—	—	—
−0.14	−0.1628	0.0344	−0.0361	−0.0713	−0.3882
−0.12	−0.1364	0.0326	−0.0257	−0.0548	−0.3171
−0.10	−0.1111	0.0296	−0.0173	−0.0407	−0.2518
−0.08	−0.0870	0.0255	−0.0108	−0.0290	−0.1924
−0.06	−0.0638	0.0205	−0.0059	−0.0190	−0.1376
−0.04	−0.0417	0.0145	−0.0026	−0.0111	−0.0877
−0.02	−0.0204	0.0076	−0.0006	−0.0047	−0.0418
(0.0000)	0.0000	0.0000	(0.0000)	0.0000	0.0000
0.02	0.0196	−0.0084	−0.0006	0.0033	0.0382
0.04	0.0385	−0.0173	−0.0022	0.0053	0.0733
0.06	0.0566	−0.0270	−0.0050	0.0060	0.1050
(0.0627)	0.0590	—	—	(0.0060)	—
0.08	0.0741	−0.0372	−0.0086	0.0056	0.1341
0.10	0.0909	−0.0479	−0.0132	0.0041	0.1603
0.12	0.1071	−0.0592	−0.0186	0.0016	0.1841
0.14	0.1228	−0.0708	−0.0247	−0.0017	0.2058
0.16	0.1379	−0.0830	−0.0316	−0.0059	0.2252
0.18	0.1526	−0.0955	−0.0391	−0.0110	0.2426
0.20	0.1667	−0.1083	−0.0472	−0.0166	0.2584
0.30	0.2308	−0.1775	−0.0958	−0.0550	0.3125
0.40	0.2857	−0.2531	−0.1551	−0.1061	0.3347
(0.4423)	0.3066	—	—	—	(0.3366)
0.50	0.3333	−0.3333	−0.2222	−0.1667	0.3333

we may analyse the phenomenon as follows: high water attending upon a low pressure amplifies as $m \to 1-0$, and after crossing $m=1$ constitutes the free wave behind the low pressure, this in its turn becomes higher and higher as it ascends an inclined bottom. On the other hand, the low pressure, running ahead of the old accompanying waves, is now in the domain $m>1$, so that it gives rise to low water this time and the surplus water produced in this way is left behind it. Thus a depression generates high water in the resonance field for these two reasons. Evidently the tide would be much higher, as the first attendant wave is higher and the second low water is deeper. In other words, we may infer that the resonance effect would be more pronounced when the almost resonant state prevails over some distance lying both in front and in the rear of the resonance point itself, and accordingly when the transfer from subcritical state to supercritical is achieved through gradual inclination of the bottom. This quasi-stationary view point will be supported by some computations of dynamical equations in the next section.

The curves in FIGURE 2 give η, the distribution of surface elevation corresponding to TABLE 1 and FIGURE 1, for the cases when

$$\left.\begin{aligned} P &= -P_0\left\{1+\cos\left(\pi\frac{x-Vt}{L_1}\right)\right\} \quad \text{for} \quad \left|\frac{x-Vt}{L_1}\right| \leq 1, \\ &= 0 \qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{aligned}\right\} \tag{15}$$

where the following constants are employed:

$$P_0 = 0.10\,\text{m}, \quad L_1 = 10\,\text{km}, \quad V = 15.3362\,\text{m/sec}\,; \tag{16}$$

$$\left.\begin{aligned} &h_0 = 24\,\text{m}, \quad h_1 = h_0(1+\lambda) = 40\,\text{m}, \\ \text{and} \quad &h_2 = h_0(1-\lambda) = 8\,\text{m}. \end{aligned}\right\} \tag{16'}$$

Namely, in terms of $g = 9.80\,\text{m/sec}^2$, we have

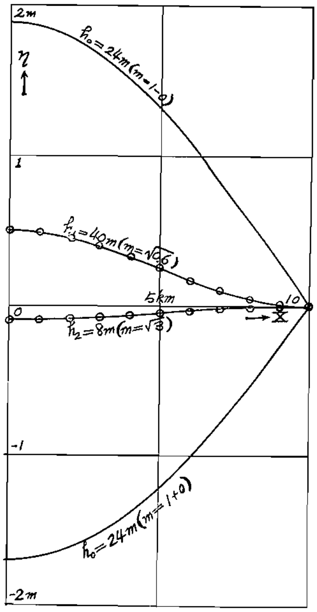

Fig. 2. Steady attendant tides.

$$\frac{V^2}{gh_0} = 1.00, \quad \frac{V^2}{gh_1} = 0.60, \quad \frac{V^2}{gh_2} = 3.00$$

The high, and the low, tides at the resonance depth are well visualized in this figure, where the surface elevation is expressed in dimensional form. A part of the numerical values are reproduced in TABLE 2, because they will be necessary later.

As stated before, special attention is paid to the linear approximation in this note, and its accuracy is examined for the stationary attendant wave. The linearly approximate form is nothing but the first terms of the right-hand sides of both of these equations (13) when developed as the series of η_1: we obtain accordingly

$$p_1 = (m^2-1)\eta_1, \quad \text{and} \quad u_0 = \eta_1. \tag{17}$$

If, therefore, we draw their curves additionally in FIGURE 1, evidently they will be found as the straight lines touching at the origin tangentially to each of the original curves. So that as long as $|p_1|$ remains very small, the expression of (17) is satisfactorily correct except the case m is close to 1. Or else by taking account of the second term in the expansion formula of p_1, we may write

$$p_1 = (m^2-1)\eta_1 - \frac{3}{2}m^2\eta_1^2, \quad \text{and} \quad u_0 = \eta_1\,; \tag{17'}$$

they may be regarded as the approximate expression, valid even if m is equal to 1. (The linearity, however, is preserved no longer!) The small circles in

TABLE 2.
Steady attendant tides.

	h=40m ($m=\sqrt{0.6}$)			h=8m($m=\sqrt{3}$)		
	nonlinear		linear	nonlinear		linear
X (km)	u (m/sec)	η (m)	η (m)	u (m/sec)	η (m)	η (m)
0.0	0.1844	0.487	0.5000	−0.1888	−0.0973	−0.1000
0.5	0.1833	0.484	0.4969	−0.1879	−0.0967	−0.0994
1.0	0.1801	0.475	0.4878	−0.1842	−0.0950	−0.0976
1.5	0.1747	0.461	0.4728	−0.1787	−0.0922	−0.0946
2.0	0.1675	0.442	0.4523	−0.1710	−0.0882	−0.0905
2.5	0.1583	0.417	0.4268	−0.1615	−0.0834	−0.0854
3.0	0.1475	0.388	0.3969	−0.1504	−0.0777	−0.0794
3.5	0.1354	0.356	0.3635	−0.1379	−0.0713	−0.0727
4.0	0.1223	0.322	0.3273	−0.1242	−0.0701	−0.0655
4.5	0.1084	0.285	0.2891	−0.1087	−0.0563	−0.0578
5.0	0.0939	0.246	0.2500	−0.0950	−0.0493	−0.0500
5.5	0.0795	0.208	0.2109	−0.0804	−0.0417	−0.0422
6.0	0.0654	0.171	0.1727	−0.0659	−0.0342	−0.0345
6.5	0.0517	0.135	0.1365	−0.0521	−0.0271	−0.0273
7.0	0.0392	0.102	0.1031	−0.0394	−0.0205	−0.0206
7.5	0.0279	0.073	0.0732	−0.0279	−0.0146	−0.0146
8.0	0.0184	0.048	0.0477	−0.0183	−0.0095	−0.0095
8.5	0.0104	0.027	0.0272	−0.0104	−0.0054	−0.0054
9.0	0.0046	0.012	0.0122	−0.0048	−0.0025	−0.0024
9.5	0.0012	0.003	0.0031	−0.0012	−0.0006	−0.0006
10.0	0.0000	0.000	0.0000	0.0000	0.0000	0.0000

FIGURE 2 show the computational results from linearizing procedure; we may conclude the accuracy is fairly good provided m is not close to 1.

4. Surge intensified by resonant bottom slope

In the preceding section, the transient phenomenon of a disturbance traveling over an inclined bottom with velocity V was discussed in terms of the characteristics of the stationary attendant wave, the results derived there having of course significance only qualitative. In this section, therefore, we attempt to integrate exactly the dynamical equations (1) and (2). In doing so, (15) together with (16) are employed as the expression for the disturbance. Again as $h(x)$, the depth of water, a slope of constant inclination connecting a uniform depth $h_1=40$ m with another one $h=8$ m is assumed; see FIGURE 3(A):

$$\left.\begin{array}{l} h(x)=24+1.60(20-x) \\ (h \text{ in m}, \quad x \text{ in km}, \quad \text{and} \quad 10<x<30). \end{array}\right\} \tag{18}$$

This is taken as an example of bottom slopes usually experienced, and the

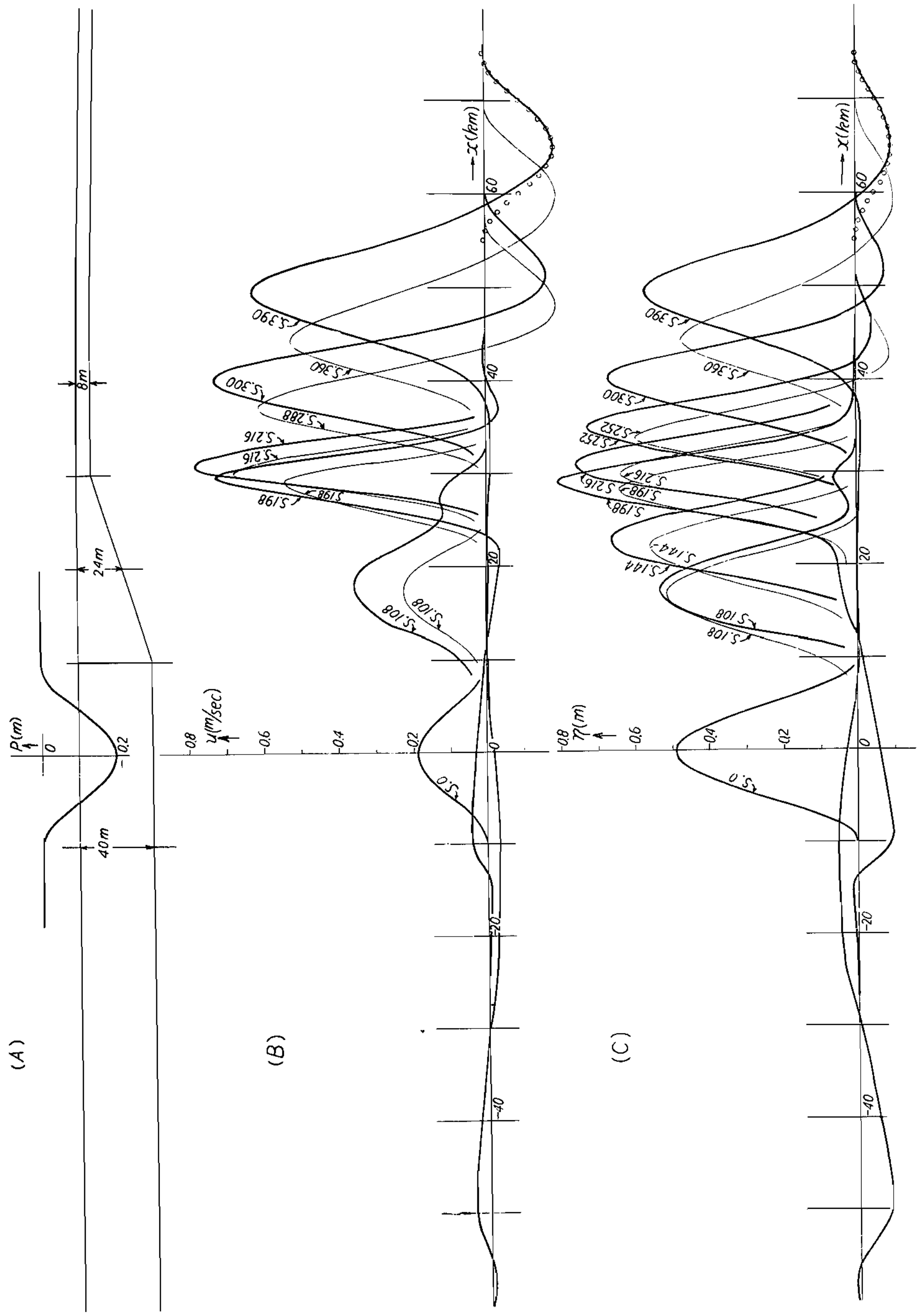

Fig. 3. Inclined bottom and resonance high water ; thick lines non-linear and thin lines linear approximations.

shallow-water-side is not supposed to be brought to an end abruptly by the seashore lest the reflected waves from there should make the situation unnecessarily complicated. Besides we need to be careful of the specification of the initial condition. We should substitute for it the values from the stationary high water on the uniform depth in front of the inclination: (u, η) computed from non-linear equations; see the column of $h=40$ m, TABLE 2, because the given tide has to react against the influence only from the bottom slope and to deform itself accordingly. Thus the values of P, u, and η at $t=0$ are shown in FIGURE 3 by (A) and by the zeroth step (thick lines S. 0) of (B) and (C), respectively.

Since it is impossible to treat equations (1) and (2) analytically under the conditions stated above, an alternative is taken which is the numerical integration with the aid of an electronic computer. As approximation by means of difference equations pertinent to our present problem, the staggered net scheme due to J. J. STOKER (8) is employed, the outline of which is as follows: introducing the speed of advance of a tidal wave, c say, by the relation

$$c^2=g(h+\eta), \tag{19}$$

and substituting c for η in equations (1) and (2), we are led to the result

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+2c\frac{\partial c}{\partial x}+E=0, \quad E\equiv g\frac{\partial}{\partial x}(P-h), \\ \text{and}\qquad &2\frac{\partial c}{\partial x}+c\frac{\partial u}{\partial x}+2u\frac{\partial c}{\partial x}=0. \end{aligned}\right\} \tag{20}$$

Now, let us suppose that at time t the values of u and c be given on the mesh points apart from each other by the distance Δx (the difference in space); see FIGURE 4. In computing their values on the next step, namely the values of u and c at the instant Δt (the difference in time) later, we choose the points where these values are to be evaluated at the positions midway of the previous mesh points. For example, in order to find the values at the position P (see the figure), u_P and η_P say, we make use of the values u_L, c_L, and u_R, c_R at the space-points L and R lying to the left, and the right, of the point P together with the value of the known function $E(x, t)$ at the midpoint M (coincident with P spatially) of the segment LR, E_M say. In other words, if we approximate the differental equations (20) through the difference equations, in which the advancing, and the central, differences are used with respect to time and space respectively, we obtain

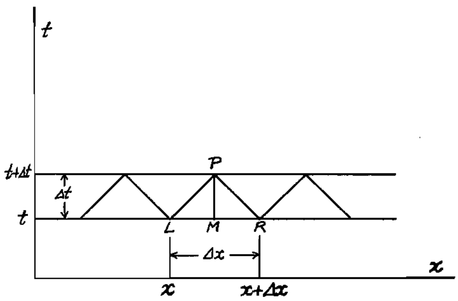

Fig. 4. A staggered net scheme.

$$\left.\begin{aligned} &\frac{u_P-u_M}{\Delta t}-u_M\frac{u_L-u_R}{\Delta x}-2c_M\frac{c_L-c_R}{\Delta x}+E_M=0, \\ \text{and}\quad &\frac{c_P-c_M}{\Delta t}-\frac{1}{2}c_M\frac{u_L-u_R}{\Delta x}-u_M\frac{c_L-c_R}{\Delta x}=0. \end{aligned}\right\} \tag{21}$$

Solving the equations with regard to the point P, we have readily

$$\left.\begin{aligned} &u_P=u_M+\frac{\Delta t}{\Delta x}\{2c_M(c_L-c_R)+u_M(u_L-u_R)-\Delta xE_M\}, \\ \text{and}\quad &c_P=c_M+\frac{1}{2}\frac{\Delta t}{\Delta x}\{2u_M(c_L-c_R)+c_M(u_L-u_R)\}, \end{aligned}\right\} \tag{21'}$$

where $\Delta xE_M=\Delta xgP_0\frac{\pi}{L_1}\left\langle \sin\frac{\pi}{L_1}(x-Vt)\right\rangle_M-\Delta xgh'(x)$

$$=0.153938\left\langle \sin\frac{\pi}{L_1}(x-Vt)\right\rangle_M+7.84000\,\delta(10,30\,\text{km})\quad \text{m}^2/\text{sec}^2. \tag{22}$$

$<\sin\phi>_M$ signifies zero in the region $|\phi_M|>\pi$, and $\delta(10,\ 30\,\text{km})$ denotes such a function as takes the value 1 when the point M is situated within the interval $x(10,\ 30\,\text{km})$, 1/2 when M lies on either of its ends, and 0 otherwise. We put $\Delta x=500$m throughout the followings.

Next, although c_M and u_M may be approximated roughly by

$$c_m\equiv(c_L+c_R)/2,\quad \text{and}\quad u_m\equiv(u_L+u_R)/2,$$

respectively, there is a calculable disagreement on the side of c, which becomes marked especially at the connection of the uniform depth with the inclined bottom, so the error is corrected throughout the whole sloping domain[1].

(i) The junction from the uniform depth to the slope: Using

$$h_L=h_1,\quad h_M=h_1,$$

and

$$h_R=h_1-\Delta h\quad (\Delta h=0.4\ \text{m}),$$

we may write

$$c_L=\sqrt{g(h_1+\eta_L)},\quad c_M=\sqrt{g(h_1+\eta_M)},$$

and

$$c_R=\sqrt{g(h_1-\Delta h+\eta_R)}.$$

We have accordingly in the first place

$$\begin{aligned} c_m&=\frac{1}{2}(c_L+c_R) \\ &\doteqdot\sqrt{gh_1}\left[1+\frac{\eta_L+\eta_R-\Delta h}{4h_1}-\frac{1}{16h_1^2}\{\eta_L^2+(\eta_R-\Delta h)^2\}\right], \end{aligned}$$

and introducing the approximate relation $\eta_m=\eta_M$ into this expression, we are led to the result

$$c_m=\sqrt{gh_1}\left\{1+\frac{\eta_M}{2h_1}-\frac{\Delta h}{4h_1}-\frac{1}{8h_1^2}\left(\eta_M^2-\Delta h\eta_M+\epsilon^2+\Delta h\epsilon+\frac{1}{2}\Delta h^2\right)\right\},$$

where we put

$$\epsilon=(\eta_L-\eta_R)/2$$

1) The computational scheme in which c is used in place of η must be criticized as unnecessary complication because of need of this correction. We should have employed η itself; see the next section.

for short. On the other hand,

$$c_M=\sqrt{gh_1\left(1+\frac{\eta_M}{h_1}\right)}=\sqrt{gh_1}\left(1+\frac{\eta_M}{2h_1}-\frac{\eta_M^2}{8h_1^2}\right),$$

and in comparing it with c_m, if we consider the fact that η_M varies indefinitely between positive and negative values while ϵ is a small quantity of higher order than Δh, we have finally

$$c_M \doteq c_m+\sqrt{gh_1}\left(\frac{\Delta h}{4h_1}+\frac{\Delta h^2}{16h_1^2}\right)=c_m+0.04962 \text{ m/sec.} \tag{23}$$

(ii) Points on the slope: Since we have

$$h_L+\eta_L=h_M+\Delta h+\eta_L, \quad \text{and} \quad h_R+\eta_R=h_M-\Delta h+\eta_R,$$

using these values we are able to compute c_L and c_R in order to derive c_m. On the other hand,

$$c_M=\sqrt{g(h_M+\eta_M)},$$

and expanding both of these relations up to the second order and comparing them with each other, we may arrive at the following result without difficulty:

$$c_M \doteq c_m+\sqrt{gh_M}\frac{\Delta h^2}{8h_M^2}=c_m+0.02\frac{\sqrt{gh_M}}{h_M^2} \text{ m/sec.} \tag{24}$$

(iii) The junction from the slope to the other uniform depth: Likewise as in (i) we have

$$c_M \doteq c_m-\sqrt{gh_2}\left(\frac{\Delta h}{4h_2}-\frac{\Delta h^2}{16h_2^2}\right)=c_m-0.10930 \text{ m/sec.} \tag{25}$$

Turning now to the problem of mesh length, we have taken $\Delta x=500$ m. Corresponding Δt must be sufficiently small and satisfy the requirement

$$\frac{\Delta x}{\Delta t}>\sqrt{2gh_1}\,; \tag{26}$$

in fact this is the condition that the computational instability migh be avoided in numerical intergation of a linear wave equation. We may not expect that this relation as it is could be the necessary and sufficient condition also for the system of the non-linear difference equations (21′), but from the conjecture that this may be still the necessary condition at least, we choose Δt within this range. What is requested further about Δt is that we should take it appropriately in order to prevent the numerical tables of $<\sin \pi(x-Vt)/L_1>_M$ from becoming too complicated. With this end in view, Δt is determined in such a way that the time taken by the disturbance to traverse the distance Δx shall be exactly in three steps (namely $3\Delta t$):

$$\left.\begin{aligned} &\Delta t=10.8675 \text{ sec.} \\ \text{and} \qquad &\frac{\Delta t}{\Delta x}=\frac{1}{3V}=0.0217350 \text{ sec/m.}\end{aligned}\right\} \tag{27}$$

Then the table of $\sin(\pi n/120)$ ($n=-120\sim120$, an integer) proves to suffice for our computational works.

Since the steady attendant wave at the depth 40 m should be substituted for the initial state, use is made of u and c transformed from u and η shown in in the second, and the third, columns of TABLE 2. As time goes on, fluctuations of water surface propagate in both directions, and the number of mesh points appearing in the computation increases accordingly. NEAC-2101, the electronic computer installed in Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, was mainly put to use. We could not help operating the machine for a long time because of its low speed of calculation.

Results of our computations are shown in FIGURE 3. The thick lines in (B) and (C) show u, the speed of stream, together with η, the surface elevation, at appropriate steps of calculations, i.e. at appropriate time-points. It is concluded that the maximum tide and the maximum velocity of flow take place, when the disturbance approaches the upper end of the slope and that the increases are 60% in elevation and 300% in velocity. Taking account of their correlation, we might say that the enormous increase in velocity seems alleviating the rise of water surface. Nevertheless, the ascent of tidal level by 60% cannot be neglected by any means. This effect should be regarded as the synthesis of results both from the sloping bottom and the resonant depth, but the discrimination is only conceptional and in reality they are inseparable and must by regarded as a single effect.

With a special view to scrutinizing the way in which high water is deformed before and behind its central portion, detailed computations were carried out throughout the whole domain of influence at the expense of considerable time involved. As supposed beforehand, in front of the disturbance low tide for the shallow depth develops gradually into the stationary attendant wave (marked by ○ in FIGURE 3). Reflected waves to the rear may practically be neglected, and the swell of water is observed to advance as a free wave, deforming its crest to be more flattened and at the same time occupying a wider range. The phenomenon that high water surging upon the shore takes the form of a bore is frequently observed, and the free tide produced in this way seems to be its main cause. Of course, the calculations performed in this section are concerned solely with a pressure disturbance, and might be different more or less from those corresponding to the overall effects of a storm. However, the fact that a storm as a whole produces the same type of influence upon the surface of water will be shown in Part II to be published soon.

5. Critical review of linear approximation

One of the problems in which we are most interested is, 'Is the linearization permissible?' To be more precise, considering the fact that the linear approximation to the attendant wave breaks down, we should examine closely the question 'To what approximation the transient resonance may be reproduced by means of a linearized model?' This study aquires increasing importance, because, according to the opinion of the present writers, it is the essential requirement, for the time being at least, especially for estimating high water due to a typhoon, that we should be allowed to cover the whole situation consistently with the linear theory.

Linearizing the fundamental equations (1) and (2), we obtain

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}=-g\frac{\partial \eta}{\partial x}-g\frac{\partial P}{\partial x},\\ \text{and}\quad &\frac{\partial \eta}{\partial t}=-\frac{\partial}{\partial x}(hu).\end{aligned}\right\}\tag{28}$$

By transforming u into λ defined by

$$h(x)u/V=\lambda(x,\ t),$$

and with the aid of the relation

$$V^2=gh_0,$$

these equations become

$$\left.\begin{aligned}&\frac{\partial \lambda}{\partial t}=-VH\left(\frac{\partial \eta}{\partial x}+G\right),\\ \text{and}\quad &\frac{\partial \eta}{\partial t}=-V\frac{\partial \lambda}{\partial x},\end{aligned}\right\}\tag{29}$$

where we write

$$G(x,\ t)=\frac{\partial P}{\partial x},\quad \text{and}\quad H(x)=h/h_0.\tag{30}$$

The steady attendant wave for the depth h_1 is found to be, as seen from (17),

$$\left.\begin{aligned}&\eta_1=p_1(m^2-1)^{-1},\quad \text{and}\quad u_0=\eta_1,\\ \text{that is}\quad &\lambda=\eta=P(m^2-1)^{-1}.\end{aligned}\right\}\tag{31}$$

Corresponding to the pattern of a disturbance (15) these values are shown in the fourth column of TABLE 2. Taking these values as the initial condition, we may proceed to integrate equation (29) numerically. Using the staggered net as in the preceding section, the appropriate difference equations are readily found to be

$$\left.\begin{aligned}&\lambda_P=\frac{1}{2}(\lambda_L+\lambda_R)+V\frac{\Delta t}{\Delta x}H_M(\eta_L-\eta_R-\Delta x G_M),\\ \text{and}\quad &\eta_P=\frac{1}{2}(\eta_L+\eta_R)+V\frac{\Delta t}{\Delta x}(\lambda_L-\lambda_R),\end{aligned}\right\}\tag{32}$$

where we write

$$\left.\begin{aligned}V\frac{\Delta t}{\Delta x}=\frac{1}{3},\quad \Delta x G_M&=\Delta x P_0\frac{\pi}{L_1}\left\langle \sin\frac{\pi}{L_1}(x-Vt)\right\rangle_M\\ &=0.015708\left\langle \sin\frac{\pi}{L_1}(x-Vt)\right\rangle_M,\end{aligned}\right\}\tag{32'}$$

$$\left.\begin{aligned}\text{and}\quad H(x)=\frac{5}{3}(x\leqq 10\ \text{km}),\quad &\frac{7}{3}-\frac{x}{15}(10\ \text{km}\leqq x\leqq 30\ \text{km}),\\ &\frac{1}{3}(30\ \text{km}\leqq x).\end{aligned}\right\}\tag{32''}$$

Computations are commenced at the front end. Here, each of λ and η is supplemented with a zero-value at a space point $\Delta x/2$ further, and after finishing every 6 steps, the range of calculation is shifted backward by one Δx, advancing $2\Delta x$ (1 km) in total. This coincides with the speed of advance of the front of the disturbance. Since detailed state of affairs at the rear end is no longer necessary here, the calculation is brought to an end in such a way that after every 6 steps the rearmost point to be computed is pushed forward by the distance Δx (namely 500 m).

Values of η and u obtained in this manner are shown by the thin lines in FIGURE 3. Both of them are found smaller compared with the solutions yielded from the non-linear equations, although general appearances are completely similar to each other. The difference of the surface elevations is 12 cm at most, or -15% in terms of the ratio. Since relatively larger increase in surface elevation results from linearization as the slope becomes gradual, we might expect with certainty that a value of inclination exists below which the linear theory yields the tidal level exceeding that from the non-linear. For actual sea-bottoms, therefore, computational error of several percent, positive or negative, would be inevitable if we adopt a linearized model, but in order to arrive at a definite conclusion with regard to this point we have still to compare the surface elevations computed by virtue of the linear, and the non-linear, theories for various bottom conditions. The fact just clarified by this one example is that there can be a case when the linear approximation, as a procedure of numerical estimation of high water, possesses the accuracy somewhat insufficient.

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References

1) ARAKAWA, H. and YOSHITAKE, M., High water due to a passing low pressure (in Japanese), *Reports on the Typhoon MUROTO*, Central Meteorological Observatory (1935), pp. 347-361.
2) YAMADA, H., High water due to a traveling disturbance (I) (in Japanese), *Umi to Sora*, vol. 16 (1936), no. 11, pp. 353-366.
3) WADACHI, K., The typhoon surge in Osaka Bay (in Japanese), *Umi to Sora*, vol. 18 (1938), no. 12, pp. 421-438.
4) HARRIS, D. L., The effect of a moving pressure disturbance on the water level in a lake, *Meteorological Monographs*, vol. 2 (1957), no. 10, pp. 46-57.
5) PLATZMAN, G. W., A numerical computation on the surge of 26 June 1954 on Lake Michigan, *Geophysica*, vol. 6 (1958), nos. 3-4, pp. 407-438.
6) UNOKI, S. and ISOZAKI, I., Some remarks in the numerical investigations on storm surges (in Japanese), *Transactions of the 9th Conference on the Coastal Engineering* (1962), pp. 1-6.
7) WIEGEL, R. L., SNYDER, C. M., and WILLIAMS, J. E., Water gravity waves generated by a moving low pressure area, *Transactions of the American Geophysical Union*, vol. 39 (1958), pp. 224-236.
8) STOKER, J. J., *Water waves*, Interscience Publishers, Inc., New York (1957), esp. **11.5**, pp. 474-481.